Indian Statistical Institute, Bangalore.
M. Math End-semester Exam : Measure-theoretic Probability

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Max. points : 30. **Time Limit : 3 hours.**

Answer any three questions only but at least one question each from Part I and Part II needs to be answered.

Give complete proofs. Please cite/quote appropriate results from class or assignments properly. You are also allowed to use results from other problems in the question paper.

1 PART I

1. (a) State and prove the extended dominated convergence theorem by replacing convergence a.e. with convergence in measure. (**6**)

   (b) Let $(\mathbb{R}, \mathcal{F})$ be a measurable space such that all singletons are measurable. For all $x \in \mathbb{R}$, define $f_x : \mathbb{R} \to \mathbb{R}$ as $f_x(y) = \mathbf{1}[y = x]$. Show that the σ-algebra generated by the collection of functions $\{f_x : x \in \mathbb{R}\}$ is the countable co-countable σ-algebra. (**4**)

2. Let μ, ν and ρ be σ-finite measures on $(\Omega, \mathcal{F})$. Assume the Radon-Nikodym derivatives here are everywhere nonnegative and finite. Suppose that $\mu << \rho$ and $\nu << \rho$, and let $A = \{d\nu/d\rho > 0, d\mu/d\rho = 0\}$. Show that $\nu << \mu$ if and only if $\rho(A) = 0$ and in which case

$$d\nu/d\mu = 1[\, d\mu/d\rho > 0\,] \frac{d\nu/d\rho}{d\mu/d\rho}. \quad (\mathbf{10})$$

3. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and m, the Lebesgue measure.

   (a) Suppose that f is a non-negative measurable function, then show that $\int f \mathrm{d}\mu = (\mu \times m)\{(x, y) : 0 < y < f(x)\}$. (**5**)

   (b) Suppose that f is a real-valued measurable function and define *graph of the function* to be $G(f) := \{(x, y) : y = f(x)\} \subset \Omega \times \mathbb{R}$. Show that $G(f)$ is measurable with respect to the product σ-algebra and $(\mu \times m)(G(f)) = 0$. (**5**).

2 PART II

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and all the random variables are defined on this space.

   (a) Let $X_1, \ldots, X_n$ be independent exponential(λ) random variables with $\lambda \in (0, \infty)$. Define the random variable $X := \min\{X_1, \ldots, X_n\}$ and the random variable $J : \Omega \to \{1, \ldots, n\}$ as follows : $J = \min\{i \in \{1, \ldots, n\} : X = X_i\}$. Show that X is a exponential random variable and find its parameters and further show that J is a uniform random variable on $\{1, \ldots, n\}$. (**5**).

(b) Let X_n be a sequence of independent Bernoulli$(1/n)$ random variables. Under which of notions of convergence - a.s. convergence, convergence in probability, weak convergence, L^2-convergence - does the sequence of random variables converge ? (**5**).

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and all the random variables are defined on this space.

   (a) Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables with finite mean μ and finite variance σ^2. Let $S_n = \sum_{i=1}^n X_i$. Show that $S_{n^2}/n^2 \stackrel{a.s.}{\to} \mu$. (**5**)

   (b) Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables such that $P(X_1 = 1) = p = 1 - P(X_1 = -1)$ with $p \neq 1/2$. Again, let $S_n = \sum_{i=1}^n X_i$ and define the random variable $Y := \sum_{n=1}^\infty 1[\,S_n = 0\,]$. Show that $Y < \infty$ a.s. (**5**).

3. (a) Define the Binomial probability distribution $Bin(n,p)$ where $n \in \mathbf{N}, p \in [0,1]$ and the Poisson probability distribution $Poi(c)$ with mean c for $c \in \mathbb{R}_+$. (**1**).

   (b) For $k \in \mathbf{N}$, define $k^{(j)} := k!/(k-j)!$ if $k \geq j$ and 0 if $k < j$. Let $X_n \stackrel{d}{=} Bin(n, p_n), X \stackrel{d}{=} Poi(c)$ with $np_n \to c$. Compute $\mathbb{E}(X_n^{(j)}), \mathbb{E}(X^{(j)})$. (**3**)

   (c) Let $X, X_n, n \geq 1$ be defined as above. Show that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables $Y, Y_n : \Omega \to \mathbb{R}$ such that $Y \stackrel{d}{=} X$, $Y_n \stackrel{d}{=} X_n$ and $Y_n \stackrel{a.s.}{\to} Y$. (**6**).